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Numerical Optimization Methods for Packing Equal Orthogonally Oriented Ellipses in a Rectangular Domain

Sh. I. Galiev and M. S. Lisafina

Kazan National Research Technological University, ul. Karla Marksa 10, Kazan, 420111 Tatarstan, Russia
e-mail: sh.galiev@mailru, e_marie@mailru

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Abstract—Linear models are constructed for the numerical solution of the problem of packing the maximum possible number of equal ellipses of given size in a rectangular domain R. It is shown that the l_p metric can be used to determine the conditions under which ellipses with mutually orthogonal major axes (orthogonally oriented ellipses) do not intersect. In R a grid is constructed whose nodes generate a finite set T of points. It is assumed that the centers of the ellipses can be placed only at some points of T. The cases are considered when the major axes of all the ellipses are parallel to the x or y axis or the major axes of some of the ellipses are parallel to the x axis and the others, to the y axis. The problems of packing equal ellipses with centers in T are reduced to integer linear programming problems. A heuristic algorithm based on the linear models is proposed for solving the ellipse packing problems. Numerical results are presented that demonstrate the effectiveness of this approach.

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1. INTRODUCTION

Following publications of other authors, we regard the figure bounded by an ellipse curve as an ellipse, assuming that it is easy to understand from the context whether we mean the curve or the figure. The problem of packing circles, ellipses, squares, rectangles, or other figures in a plane or a given domain, for example, a square, rectangle, triangle, circle, or strip has been extensively studied and various approaches have been developed for its solution (see, for example, [1–5]). The interest in the packing problem and the conceptually related covering problem is motivated primarily by their possible applications in practice. Numerous applications of these problems in economics and engineering can be found in [6–8]. Ellipse packings are also used in the analysis of structures consisting of elliptic molecules in crystals [9], while ellipsoid packings are used to analyze structures of cement solutions [10] (for ellipse packing problems, see also [11–14]).

At present, various models of packing problems are available, in which the packing problem is often reduced to a nonlinear programming problem and, in the case of square or rectangle packing, to a linear programming problem (for the use of linear programming problems, see, e.g., [4]). There is a series of works in which circles are packed in a rectangle (the circle is a special case of an ellipse). Numerous publications are available on packings in a triangular domain, a circular domain, a strip, etc. We do not present a detailed analysis of publications concerning packing problems but indicate only some works where overviews can be found (see, e.g., [1, 2, 4, 5] and the bibliography therein). Additionally, there are studies addressing conceptually related problems of covering shapes by ellipses (see [15, 16]).

Note that the problem of packing figures (for example, circles) is usually stated as packing n circles of maximum possible (previously unknown) diameter in a given domain. In applications, as a rule, the sizes of the figures to be packed are known and the task is to put the maximum possible number of figures in a given domain. We follow the approach to the packing problem when equal ellipses to be packed have given sizes (parameters). Moreover, we consider the case when the major axes of all the ellipses are parallel to the x or y axis or when some ellipses have major axes parallel to the x axis, while the others, to the y axis; i.e., the ellipses are orthogonally oriented. The case of ellipse axes having relative positions other than these is not considered.

In this paper, we propose linear models for the numerical optimization of the number (determining the maximum possible number) of equal ellipses of fixed size that can be packed in a given rectangular domain R.

It is shown that the l_p metric can be used to determine the conditions under which ellipses with mutually orthogonal major axes (orthogonally oriented ellipses) do not intersect. In R a grid is constructed whose nodes generate a finite set T of points. It is assumed that the centers of the ellipses can be placed only at some points of T. The packing problems are reduced to integer linear programming problems. A heuristic algorithm based on the linear models is proposed for solving the ellipse packing problems. Numerical results are presented that demonstrate the effectiveness of this approach.

2. FORMULATION OF THE PROBLEM

Assume that a Cartesian coordinate system xOy is introduced in a plane P, and let $d(s, t)$ be the Euclidean distance between points s and t. Let E be an ellipse centered at the point $C(x, y)$ with semimajor and semiminor axes a and b, respectively. Here, a and b are assumed to be parameters of the ellipse. If the major axis of an ellipse is parallel to the x axis, then the ellipse with an indicated center $C(x, y)$ and parameters a and b is denoted by $E(X; x, y, a, b)$. If the major axis of an ellipse is parallel to the y axis and its center and parameters are the same as before, then the ellipse is denoted by $E(Y; x, y, a, b)$.

In what follows, we assume that R is a closed rectangular domain in the plane P and the sides of R are parallel to the coordinate axes. A collection of m equal open ellipses E_j, $1 \le j \le m$, forms a packing in R if each ellipse E_j is contained in R, $1 \le j \le m$, and each point s from R belongs to at most one of these ellipses.

The density of a packing is defined as the ratio of the sum of the areas of all packed ellipses to the area of the figure in which they are packed. The packing density is denoted by p.

Problem Z1. Determine the maximum possible number of equal open ellipses with given parameters that can be packed in a domain R and determine the positions of their axes and centers.

Let R^* ($R^* \subset R$) be the set of all points s from R such that s is the center of an ellipse E contained in R: $E \subset R$. Assume that the set R^* is not empty. On R^* we construct a rectangular grid with a chosen step Δ in x and y. The nodes of the grid are denoted by $t_1, t_2, \ldots, t_n$, $n \ge 1$. Let $T = \{t_1, t_2, \ldots, t_n\}$, and let each point t_i, $1 \le i \le n$, belong to R^*.

A grid on R^* can be constructed in an arbitrary manner or the points of $T = \{t_1, t_2, \ldots, t_n\}$ can be specified (chosen) without using a grid or with partial use of grid nodes.

Assume that the set T has been somehow constructed.

Problem Z2. Determine the maximum possible number of equal open ellipses with given parameters a and b centered at some points of T that can be packed in R and determine the positions of their axes and centers.

In what follows, instead of solving problem Z1 on R, we solve some versions of problem Z2. Clearly, the resulting packing gives an approximate solution of problem Z1.

3. TANGENCY CONDITION FOR ELLIPSES WITH MAJOR AXES PARALLEL TO THE x OR y AXIS AND METHODS FOR GRID CONSTRUCTION

The following result can be proved to hold.

Proposition 1. *Let $E_1(X; 0, 0, a, b)$ and $E^*(X; 0, 0, 2a, 2b)$ be ellipses with indicated parameters centered at the origin. Then any ellipse $E_2(X; x_2, y_2, a, b)$ with its center $C_2(x_2, y_2)$ lying on the curve enclosing $E^*(X; 0, 0, 2a, 2b)$ has a unique intersection point with $E_1(X; 0, 0, a, b)$.*

Corollary 1. Proposition 1 remains valid if the symbol X in the ellipse notation is everywhere replaced by Y.

For brevity, ellipses with major axes parallel to the x axis are referred to as horizontal ellipses, while ellipses with major axes parallel to the y axis are referred to as vertical ellipses.

It is well known that, in the case of an optimal packing of equal circles in the entire plane, the lines passing through their centers generate a skewed grid. One of the grid lines is parallel to the x axis, and the other, to the vector $(1, \sqrt{3})$. For packing circles or ellipses in a bounded set, no results of this type are unknown to the authors. It is natural to assume that a similar skewed grid may also be optimal for ellipses arranged in a staggered manner. By a rectangular grid, we mean one with its generating lines parallel to the coordinate axes.

By applying a more detailed analysis, it can be shown that a rectangular grid is preferable to a skewed grid and that the best strategy for varying the grid step in grid construction is by halving it. Moreover, it can be established that a rectangular grid with different grid steps in the x and y axes is acceptable. Similar results hold for vertical ellipses.

Clearly, there are other methods for constructing a grid, but the indicated ones are rather convenient. In many respects, the shape of the grid depends on the domain G and, in the authors' view, there is no universal method for grid construction in the general case.

4. PACKING OF HORIZONTAL ELLIPSES

Consider ellipses with parameters a and b and with major axes parallel to the x axis. Let R_x^* be a rectangle lying in R ($R_x^* \subset R$) such that the upper and lower sides of R_x^* are separated from the corresponding sides of R by a distance of b, while the lateral sides of R_x^* are separated from the corresponding lateral sides of R by a distance of a. Obviously, R_x^* is the set of all points $s(x, y)$ from R such that $s(x, y)$ is the center of an ellipse $E(X; x, y, a, b)$ contained in R: $E(X; x, y, a, b) \subset R$.

On R_x^* we construct a rectangular grid, whose nodes are denoted by $t_{x1}, t_{x2}, \ldots, t_{xn}$, $n \geq 1$. Let $T_x = (t_{x1}, t_{x2}, \ldots, t_{xn})$, and let each point t_{xi}, $1 \leq i \leq n$, belong to R_x^*. Now we consider problem Z2 assuming that the major axes of the ellipses are parallel to the x axis and the centers of the ellipses can be placed only at some points of T_x.

Let C_i be the center of the ith ellipse, $1 \leq i \leq N_x$, $1 \leq N_x \leq n$, where N_x is the number of ellipses to be packed. Following a well-known method, we introduce the variables

$$z_i = \begin{cases} 1 & \text{if the center of } C_i \text{ coincides with } t_{xi}, \\ 0 & \text{otherwise,} \end{cases} \quad 1 \leq i \leq n. \tag{1}$$

Let the center C_i of the ellipse $E_i(X; x_i, y_i, a, b)$ coincide with the point $t_{xi}(x_i, y_i)$; i.e., $z_i = 1$, $1 \leq i \leq n$. For E_i to be disjoint with the other ellipses to be packed, it is necessary that z_j be zero for all points (ellipse centers) t_{xj}, $i \neq j$, lying $E^*(X; x_i, y_i, 2a, 2b)$. Suppose that there are r_i points t_{xj}, $i \neq j$, $1 \leq j \leq n$, that lie inside $E^*(X; x_i, y_i, 2a, 2b)$. Denote by $\mathrm{int}(X)$ the interior of the set X.

The above nonintersection condition for horizontal ellipses can be written as follows:

$$\text{if } z_i = 1, \text{ then, for all } j \text{ such that } t_{xj} \in \mathrm{int}(E^*(X; x_i, y_i, 2a, 2b)) \text{ it is true that } z_j = 0, \quad i \neq j, \quad 1 \leq j \leq n. \tag{2}$$

The following result can be proved.

Proposition 2. *The nonintersection condition* (2) *for horizontal ellipses is equivalent to the condition*

$$r_i z_i + \sum_{j: t_{xj} \in \mathrm{int}(E^*(X;\, x_i,\, y_i,\, 2a,\, ab)),\, i \neq j} z_j \leq r_i. \tag{3}$$

The satisfaction of condition (2) (or (3)) means that, if the center of an ellipse with parameters a and b is placed at the point $t_{xi}(x_i, y_i)$, $1 \leq i \leq n$, then the center of any other ellipse can be placed only at points t_{xj}, $i \neq j$, $1 \leq j \leq n$, such that $t_{xj} \notin \mathrm{int}(E^*(X; x_i, y_i, 2a, 2b))$. Therefore, under this condition, open ellipses with parameters a and b centered at t_{xi} and t_{xj} do not intersect.

We introduce the coefficients

$$a_{ij} = \begin{cases} 1, & t_{xj} \in \mathrm{int}(E^*(X; x_i, y_i, 2a, 2b)), \\ 0, & t_{xj} \notin \mathrm{int}(E^*(X; x_i, y_i, 2a, ab)), \end{cases} \quad i \neq j, \quad 1 \leq i, j \leq n, \quad a_{ii} = r_i, \quad 1 \leq i \leq n.$$

Obviously, for given i, condition (2) and, hence, condition (3), can now be written as $a_{i1}z_1 + a_{i2}z_2 + \ldots + a_{in}z_n \leq r_i$, $1 \leq i \leq n$.

Let A be an $n \times n$ matrix with elements a_{ij}, where $1 \leq i, j \leq n$, and let Z and M_x be vectors

$$Z = (z_1, z_2, \ldots, z_n)^{\mathrm{T}}, \quad M_x = (r_1, r_2, \ldots, r_n)^{\mathrm{T}}.$$

Consider the problem

$$N_x = \sum_{i=1}^{n} z_i \longrightarrow \max \tag{4}$$

subject to the constraints

$$AZ \le M_x, \quad z_i \in \{0, 1\}, \quad 1 \le i \le n. \tag{5}$$

This problem is to maximize the number of variables z_i taking a value of 1 and such that z_i and z_j are both equal to 1 if $t_{xi} \notin \mathrm{int}(E^*(X; x_j, y_j, 2a, 2b))$ and $t_{xj} \notin \mathrm{int}(E^*(X; x_i, y_i, 2a, 2b))$. Therefore, the open ellipses centered at the points t_i and t_j do not intersect. Consequently, problem (4), (5) is to pack in G the maximum possible number of horizontal ellipses with parameters a and b centered at some points of T_x. Solving problem (4), (5), we find the number N_x of packed ellipses with parameters a and b, while the found values z_i determine the positions of the ellipses' centers.

Note that conditions (2) and (3) and the reduction of them to linear constraints were proposed in [17] for the case of circle packing.

5. THE CASE OF ELLIPSES WITH MAJOR AXES PARALLEL TO THE y AXIS

Obviously, if the major axes of all the ellipses are parallel to the y axis, then the argument is similar to the preceding case.

Instead of the rectangle R_x^*, we construct a rectangle R_y^* lying inside R ($R_y^* \subset R$) such that the upper and lower sides of R_y^* are separated from the corresponding sides of R by a distance of a, while the lateral sides of R_y^* are separated from the corresponding lateral sides of R by a distance of b. Obviously, R_y^* is the set of all points $s(x, y)$ from R such that $x(x, y)$ is the center of a vertical ellipse $E(Y; x, y, a, b)$ contained in R: $E(Y; x, y, a, b) \subset R$. On the set R_y^*, we construct a rectangular grid. The grid nodes form a set $T_y = \{t_{y1}, t_{y2}, \ldots, t_{ym}\}$, where m is the number of elements in T_y; moreover, each point t_{yi}, $1 \le i \le m$, belongs to R_y^*. Now we consider and solve problem Z2 assuming that the centers of the vertical ellipses to be packed can be placed only at some points of T_y. Let C_i be the center of an ellipse. We introduce the variable

$$v_i = \begin{cases} 1 & \text{if the center of } C_i \text{ coincides with } t_{yi}, \\ 0 & \text{otherwise}, \end{cases} \qquad 1 \le i \le m.$$

Suppose that there are s_i points t_{yj}, $i \ne j$, $1 \le j \le n$, that are inside the ellipse $E^*(Y; x_i, y_i, 2a, 2b)$. Define the coefficients

$$b_{ij} = \begin{cases} 1, & t_{yj} \in \mathrm{int}(E^*(Y; x_i, y_i, 2a, 2b)), \\ 0, & t_{yj} \notin \mathrm{int}(E^*(Y; x_i, y_i, 2a, 2b)), \end{cases} \quad i \ne j, \quad 1 \le i, j \le m, \quad a_{ii} = s_i, \quad 1 \le i \le m.$$

They generate an $m \times m$ matrix, which is denoted by B.

Let V and M_y be vectors $V = (v_1, v_2, \ldots, v_m)^{\mathrm{T}}$ and $M_y = (s_1, s_2, \ldots, s_m)^{\mathrm{T}}$. Consider the problem

$$N_y = \sum_{i=1}^{m} v_i \longrightarrow \max \tag{6}$$

subject to the constraints

$$BV \le M_y, \quad v_i \in \{0, 1\}, \quad 1 \le i \le m. \tag{7}$$

This problem is to maximize the number of variables v_i taking a value of 1 and such that v_i and v_j are both equal to 1 if $t_{yi} \notin \mathrm{int}(E^*(Y; x_j, y_j, 2a, 2b))$ and $t_{yj} \notin \mathrm{int}(E^*(Y; x_i, y_i, 2b, 2b))$. Therefore, the open vertical ellipses centered at the points t_{yi} and t_{yj} do not intersect. Therefore, problem (6), (7) is to pack in R the maximum possible number of vertical ellipses with parameters a and b centered at some points of T_y.

6. TANGENCY CONDITIONS FOR ELLIPSES WITH MUTUALLY ORTHOGONAL MAJOR AXES

Let E_1 and E_2 be ellipses whose major axes are mutually orthogonal. As is known, the determination of intersection points of arbitrary ellipses is generally reduced to solving fourth-degree equations. We try to avoid the solution of such an equation and the enumeration of its roots.

Let E_1 and E_2 be two ellipses with a tangency point F. Assume that their semimajor and semiminor axes are a and b, respectively, and their major axes are mutually orthogonal. Moreover, let $a + b$ be a constant: $a + b = \text{const}$. Let C_1 and C_2 be the centers of E_1 and E_2, respectively. If the eccentricities of the ellipses are zero ($e_1 = e_2 = 0$ $(a = b)$), i.e., E_1 and E_2 are circles, then C_2 obviously lies on the circle of radius $r = 2a$ ($r = a + b$) centered at C_1. Denote this circle by W. In the degenerate case, when the eccentricities of the ellipses are equal to unity, the center of E_2 (when E_1 and E_2 have a tangency point) obviously lies on the boundary of the square G with sides parallel to the coordinate axes and with side length $4a$. If the value of the eccentricity e is other than 0 or 1, then, as the tangency point F of the ellipses move along the boundary of E_1, the center C_2 of E_2 describes a curve S contained between W and G.

Let L be the ray starting at the center C_1 of E_1 and passing through the center C_2 of E_2, and let α be the angle between the Ox axis and L. If α varies, then, according to computations, $[d(C_1, C_2) - r]$ increases, reaching its maximum at $\alpha = \pi/4$, and then decreases to zero at $\alpha = \pi/2$; next, the pattern is repeated. Thus, the tangency point of two equal ellipses with mutually orthogonal major axes cyclically approaches W and G and moves away from them. Interestingly, similar behavior is exhibited by points of "circles" generated by the l_p metric. Following a well-known procedure, the distance between arbitrary points s and l is defined as

$$d_p(s, t) = [|x_s - x_t|^p + |y_s - y_t|^p]^{1/p}, \quad 2 \le p < \infty; \quad d_p(s, t) = \max(|x_s - x_t|, |y_s - y_t|), \quad p = \infty.$$

For $p = 2$, this is the usual Euclidean distance and $d_p(s, t) = r$ is the equation of a circle of radius r centered, for example, at the point $s(x_s, y_s)$; moreover, if $r = a + b$, then the point $t(x_t, y_t)$ lies on the circle W. For $p = \infty$, the equation $d_p(s, t) = r$ defines a square G centered at $s(x_s, y_s)$ with sides parallel to the coordinate axes and with side length $4a$. When $2 < p < \infty$, the equation $d_p(s, t) = r$ with fixed s generates curves S_p lying between W and G. Relying on this analogy, we approximate S by a curve S_p.

In the special case of $\alpha = \pi/4 + k\pi/2$, where k is an integer, the tangency point of E_1 and E_2 is fairly easy to find; for example, for $\alpha = -\pi/4$, the coordinates of the tangency point are given by

$$x = \frac{a^2 c}{a^2 + b^2}, \quad y = x - c = \frac{a^2 c}{a^2 + b^2} + \sqrt{a^2 + b^2}.$$

In cases other than those indicated above, the tangency points of the ellipses can be found as follows. Let F be a tangency point of E_1 and E_2. Given one of the coordinates of the center C_2 of E_2, say, x_2, the other coordinate of C_2 in the case of $-\pi/2 \le \alpha \le 0$ can be found by solving the nonlinear programming problem

$$y_2 \longrightarrow \min,$$

$$(x - x_2)^2/b^2 + (y - y_2)^2/a^2 = 1, \quad x^2/a^2 + y^2/b^2 = 1, \quad y_2 < 0, \quad x_2 = \text{const}, \quad 0 \le x_2 \le a + b.$$

A similar strategy is used for other values of α. As a result, we find the coordinates of tangency points for arbitrary values of α.

Now let us choose a parameter p. Choosing a position of the center C_2 of E_2 (assuming that E_2 is tangent to E_1), we draw the ray L starting at C_1 and passing through C_2. Therefore, C_2 lies on the curve S. Let C_3 be an intersection point of S_p and L lying further away from C_1 than the point C_2. The value of p is chosen so that C_3 is as close as possible to C_2; in other words, the curve S_p is as close as possible to S (on L), i.e., $d_p(C_1, C_2) \approx (a + b)$. With a prescribed accuracy with respect to p, the value of p is chosen as follows. Starting with a chosen p_0 ($p_0 \ge 2$), p is increased with the step Δp so that the ratio $q = d_p(C_1, C_2)/(a + b)$ with the new p is larger than unity, while, with the preceding value of p (equal to $p - \Delta p$), this ratio is less than or equal to unity. The resulting p is used as its desired value. It ensures that the intersection point (C_3) of the ray l and the curve S_p is farther away from C_1 than from C_2 (according to the Euclidean distance). Therefore, if the center of E_2 lies at C_3, the ellipses centered at C_1 and C_3 do not intersect. At different angles between L and the Ox axis, we can obtain different values of p. For example, at the eccentricity $e =$

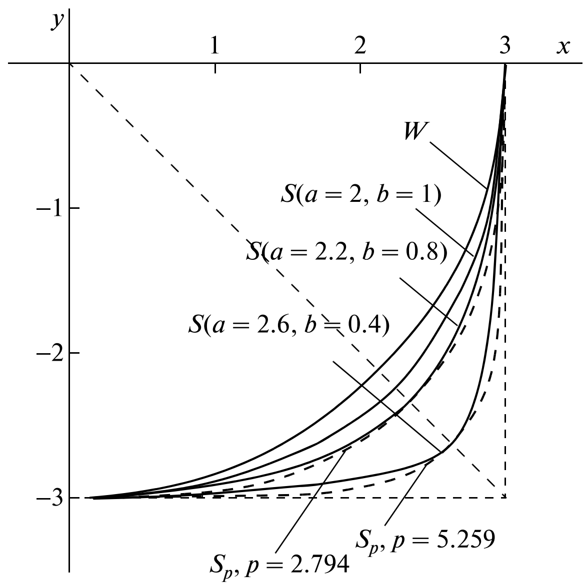

Fig. 1. Curves S and S_p.

0.86603, the parameter p for different angles α ranges from 2.145 to 2.354. Clearly, if we choose the largest value of p for various inclination angles of L, then, for each L, its intersection point with S_p (the point C_3) is father from C_1 than from C_2. Consequently, if the center of E_2 lies on S_p, the ellipses E_1 and E_2 do not intersect.

In this paper, we obtained sets of coordinates x_2 and y_2 of the point C_2 for different values of a and b. Figure 1 shows the curves S constructed for ellipses with parameters $a = 2$, $b = 1$; $a = 2.2$, $b = 0.8$; and $a = 2.6$, $b = 0.4$ and the curves S_p (dashed) in the case of $x \geq 0$ and $y \leq 0$. Additionally, the figure presents part of the circle W of radius $a + b$. When the eccentricity of the ellipse does not exceed 0.86603, the curves S and S_p nearly coincide, while in the cases, for example, $e = 0.93154$ and $e = 0.98808$, the curves S_p are some distance away from S, as can be seen in Fig. 1. The distance between S and S_p can be found as the distance between the points C_2 and C_3 for a chosen angle between L and the x axis. The longest distance for chosen ellipse parameters is denoted by γ. Table 1 gives the resulting values of p and γ for various ellipse parameters. Note that, for ellipses with a given eccentricity, p is chosen once and the resulting value of p is then used for all ellipse packing problems of this type in various rectangular domains.

7. PACKING ELLIPSES SOME OF WHICH HAVE ORTHOGONAL MAJOR AXES (ORTHOGONALLY ORIENTED ELLIPSES)

Assume that sets T_x and T_y have been constructed. Although some of the points t_{xi} and t_{yj} may coincide, we retain the double notation for them, since the coincidence of the center of an ellipse with t_{xi} means that its major axis is parallel to the x axis, while the coincidence with t_{yj} means that its major axis is parallel to the y axis.

Table 1. Values of the parameter p

Parameters a/b	Eccentricity e	Parameter p	γ
1.6/1.4	0.48412	2.01300	0.00200
1.8/1.2	0.74536	2.1200	0.00722
2/1	0.86603	2.35900	0.01957
2.2/0.8	0.93154	2.79500	0.03813
2.4/0.6	0.96825	3.59500	0.06007
2.6/0.4	0.98809	5.27500	0.07942

For ellipse packing, we have to take into account the nonintersection condition for ellipses with major axes parallel to the x axis. This condition is written as $AZ \le M_x$ (see (5)). Then we take into account the nonintersection conditions for ellipses with major axes parallel to the y axis: $BV \le M_y$ (see (7)). Next, the nonintersection conditions for ellipses with mutually orthogonal major axes are taken into account.

The nonintersection conditions for horizontal and vertical ellipses are written in the form of conditions similar to (3). Each horizontal ellipse cannot intersect each vertical ellipse. In Section 4, we used the fact that the center $t_i(x_i, y_i)$ of the ellipse $E(X, x_i, y_i, a, b)$ is not an interior point of the ellipse $E(X, x_j, y_j, 2a, 2b)$ centered at $t_j(x_j, y_j)$, while now we use the distance $d_p(t_i, t_j)$ with a chosen parameter p. The conditions under which vertical ellipses do not intersect horizontal ones are written in a similar manner.

Let the center of an ellipse E_i be at the point t_{xi}; i.e., this is a horizontal ellipse. Assume that t_{yj} is the center of a vertical ellipse E_j. According to Section 6, these ellipses do not intersect if the distance between t_{xi} and t_{yj} based on the found p-metric is at least $a + b$.

Suppose that, for t_{xi}, there are q_i points t_{yj} for which $d_p(t_{xi}, t_{yj}) < a + b$, $1 \le i \le n$. The coefficients c_{ij} are defined in a similar fashion to the coefficients a_{ij} for constraints (5):

$$c_{ij} = \begin{cases} 1, & d_p(t_{xi}, t_{yj}) < a + b, \\ 0 & \text{otherwise}, \end{cases} \quad i = 1, \dots, n, \quad j = 1, \dots, m, \tag{8}$$

$$c_{i,\, i+m} = q_i, \quad 1 \le i \le n, \quad c_{ij} = 0, \quad 1 \le i \le n, \quad m + 1 \le j \le m + n, \quad j \ne i + m.$$

Remark 1. The determination of the coefficients c_{ij} assumes that p is known (determined). When the ellipse eccentricity is, for example, 0.86603, the distance between the centers of the ellipses measured along the line L differs from $d_p(t_{xi}, t_{yj})$ by at most $\gamma = 0.02$ (see Table 1). If this accuracy (γ) is satisfactory, then p can be set to a value from Table 1. Otherwise, for each slope of the line L (for pairs of points t_{yj} and t_{xi}), its own value of p has to be chosen.

From the coefficients c_{ij}, $1 \le i \le n$, $m + 1 \le j \le m + n$, we can construct an $n \times (m + n)$ matrix C. Define the vectors $VZ = (v_1, \dots, v_m, z_1, \dots, z_n)^{\mathrm{T}}$ and $M_{vz} = (q_1, \dots, q_n)^{\mathrm{T}}$. The constraints are constructed taking into account that some of c_{ij} vanish (see (8)):

$$\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} & c_{1,\, m+1} & 0 & \dots & 0 \\ c_{21} & c_{22} & \dots & c_{2m} & 0 & c_{2,\, m+2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nm} & 0 & 0 & \dots & c_{n,\, n+m} \end{pmatrix} \begin{pmatrix} v_1 \\ \dots \\ v_m \\ z_1 \\ \dots \\ z_n \end{pmatrix} \le \begin{pmatrix} q_1 \\ q_2 \\ \dots \\ q_n \end{pmatrix}. \tag{9}$$

Inequalities (9) are the conditions under which any possible horizontal ellipse does not intersect any possible vertical ellipse.

Now suppose that the center of an ellipse E_i coincides with the point t_{yi}. Let us derive the conditions under which E_i does not intersect ellipses E_j with major axes being orthogonal to that of E_i. For this purpose, we again use the distance $d_p(t_{yi}, t_{xj})$. Suppose that, for the point t_{yi}, there are l_i points t_{xj} for which $d_p(t_{yi}, t_{xj}) < a + b$, $1 \le i \le m$. Define the coefficients

$$h_{ij} = \begin{cases} 1, & d_p(t_{yi}, t_{xj}) < a + b, \\ 0 & \text{otherwise}, \end{cases} \quad i = 1, \dots, m, \quad j = 1, \dots, n, \tag{10}$$

$$h_{i,\, i+n} = l_i, \quad 1 \le i \le m, \quad h_{ij} = 0, \quad 1 \le i \le m, \quad n + 1 \le j \le n + m, \quad j \ne i + n.$$

When determining h_{ij}, we have to take into account Remark 1. These coefficients $m \times (n + m)$ are used to construct a matrix H. Define the vectors $ZV = (z_1, \dots, z_n, v_1, \dots, v_m)^{\mathrm{T}}$ and $M_{zv} = (l_1, \dots, l_m)^{\mathrm{T}}$.

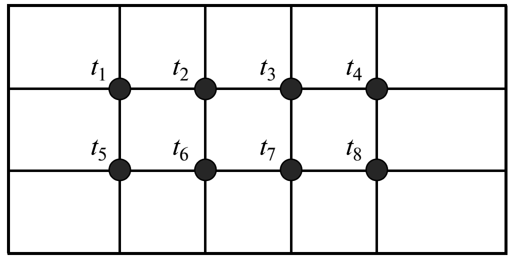

Fig. 2. Introduction of weights of the levels.

The constraints are constructed taking into account the fact that some of h_{ij} vanish (see (10)):

$$\begin{pmatrix} h_{11} & h_{12} & \dots & h_{1n} & h_{1,n+1} & 0 & \dots & 0 \\ h_{21} & h_{22} & \dots & h_{2n} & 0 & h_{2,m+2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ h_{m1} & h_{m2} & \dots & h_{m,n} & 0 & 0 & \dots & h_{m,m+n} \end{pmatrix} \begin{pmatrix} z_1 \\ \dots \\ z_n \\ v_1 \\ \dots \\ v_m \end{pmatrix} \le \begin{pmatrix} l_1 \\ l_2 \\ \dots \\ l_m \end{pmatrix}. \tag{11}$$

Inequalities (11) are the conditions under which any possible vertical ellipse does not intersect any possible horizontal ellipse. Clearly, if conditions (9) hold, then conditions (11) hold as well and vice versa. The choice of conditions (9) or (11) depends on the number of their rows, if we want to reduce the number of constraints.

Now the packing problem for ellipses some of them have major axes parallel to the x axis, while the others, to the y axis, has the form

$$z_1 + \dots + z_n + v_1 + \dots + v_m \longrightarrow \max \tag{12}$$

subject to the constraints

$$\begin{gathered} AZ \le M_x, \quad BV \le M_y, \\ C(VZ) \le M_{vz} \quad \text{or} \quad H(ZV) \le M_{zv}, \\ z_i \in \{0, 1\}, \quad 1 \le i \le n, \quad v_j \in \{0, 1\}, \quad 1 \le j \le m. \end{gathered} \tag{13}$$

8. ALGORITHM FOR PACKING ELLIPSES IN A GIVEN DOMAIN

Problems (4), (5); (6), (7); and (12), (13) are integer linear programming problems. Obviously, they can be solved using any method (algorithm) for integer linear programming problems if it can be implemented in an acceptable time. For high dimensions (large numbers of variables), the solution of such problems by applying well-known methods is too expensive (frequently unacceptably) in terms of CPU time. For this reason, we propose a heuristic algorithm based on a natural stage-by-stage packing procedure. Specifically, the algorithm first packs a few (possibly, three or four) layers of ellipses. Then another few are packed taking into account the constructed packing, etc.

Before describing this heuristic algorithm, we present some auxiliary arguments. Consider the problem of packing the maximum possible number of open equal horizontal ellipses with parameters $a = 0.45$ and $b = 0.3$ into a rectangle R of width 1.8 and height 0.9. Obviously, two such ellipses can be packed in R. This solution can be obtained by constructing a model of form (4), (5). Let the step for grid construction in x and y be $\Delta = 0.3$. Then the set T consists of eight points: $T = \{t_1, t_2, \dots, t_8\}$ (see Fig. 2). Problem (4), (5) is written as

$$N_1 = z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 \longrightarrow \max \tag{14}$$

with constraints (5) constructed for the above set T and chosen ellipses parameters. Solving this problem, we find that the largest N_1 is equal to 2 and the centers of the ellipses can be placed at the points t_1 and t_4, t_1 and t_8, t_4 and t_5, or t_5 and t_8.

If we want the packed ellipses to be as low as possible (in R), the model has to be changed. Specifically, we assume that the points t_1, t_2, t_3, and t_4 are at the second level, while the points t_5, t_6, t_7, and t_8 are at the first level. The levels are indexed from bottom to top. Points placed at the same level have identical coordinates y. We want the centers of the ellipses to be as low as possible. For this reason, the variables corresponding to the points at the first level are multiplied by a constant (weight) c_1 that is larger than the weight c_2 used to multiply the variables at the second level. Let $c_1 = 2$ and $c_2 = 1$. Then instead of (14), we write

$$N_2 = z_1 + z_2 + z_3 + z_4 + 2z_5 + 2z_6 + 2z_7 + 2z_8 \longrightarrow \max. \tag{15}$$

Solving problem (15) with the same constraints as above, we find that the largest value N_2 is 4 and the centers of two ellipses can be placed only at the points t_5 and t_8. Thus, by choosing weights of the levels, we ensured that the centers of the ellipses are as low as possible. Clearly, a similar pressing down procedure can be applied to other rectangles and other ellipses, since the constraints of the problem ensure that the ellipses do not intersect and, if an ellipse can be placed lower, then the objective function value increases; i.e., due to the level weights introduced, the ellipses are shifted down whenever possible.

Now we find out the effect that can be achieved by introducing certain subsets of the set R.

First, consider the case of packing horizontal ellipses. Suppose that the coordinates y of points from R satisfy the condition $\alpha \le y \le \beta$, while the coordinates y of points from R^* satisfy $\alpha^* \le y \le \beta^*$; here, R^* is the set of possible positions of the ellipse centers in R. Let α_1 be chosen so that $\alpha_1 \ge \alpha^* + 4r$ and $\alpha_1 \le \beta$, where r is the semiaxis parallel to the y axis: for horizontal ellipses, $r = b$ (while for vertical ellipses, $r = a$). The set $D_{\alpha 1}$ is defined as the subset of R that lies below the line $y = \alpha_1$. For $D_{\alpha 1}$, we construct $D^*_{\alpha 1}$ in a similar manner to the construction of the set R^* for R: $D^*_{\alpha 1}$ contains a point s from $D_{\alpha 1}$ if and only if the open ellipse centered at s is contained in $D_{\alpha 1}$. Next, on $D^*_{\alpha 1}$ we construct a grid whose nodes generate the set $T_{\alpha 1} = \{t_1, t_2, \ldots, t_{n1}\}$. Points t_j with identical coordinates y_j, $1 \le j \le n1$, are thought of as lying at the same level. The set $T_{\alpha 1} = \{t_1, t_2, \ldots, t_{n1}\}$ constructed for $D^*_{\alpha 1}$ may contain points lying at k levels, where $k \ge 1$ and the levels are indexed from bottom to top. We introduce k weights of these levels: $c_1 \ge c_2 \ge \ldots \ge c_k$, where c_j is the weight of the jth level. In the objective function, the variables z_j are multiplied by the weight of the level that contains the point t_j, $1 \le j \le n1$. As a result, we obtain a new objective function, while the constraints are constructed in the same way as for problem (4), (5) but with the set $D_{\alpha 1}$. The resulting problem with allowance for the level weights is regarded as an auxiliary problem, and its solution is regarded as a solution for the set $D_{\alpha 1}$. As a result, for $D_{\alpha 1}$, we obtain an ellipse packing in $D_{\alpha 1}$, $D_{\alpha 1} \subseteq R$, which is denoted by $P_{\alpha 1}$.

Suppose that $D_{\alpha(i-1)}$ has been constructed and a packing $P_{\alpha(i-1)}$ has been found on it. Let Δ be the step for grid construction in y. The boundary of the set $D_{\alpha i}$ ($i \ge 2$) consists of

- the segments of the lines $y = \alpha_i$ and $y = \alpha_{i-1} - 2r - \Delta$, contained in R;
- the portions of the boundary of R lying between these lines;
- if, above the line $y = \alpha_{i-1}$, there is no similar line, then the boundary of $D_{\alpha i}$ consists of the segment of the line $y = \alpha_{i-1} - 2r - \Delta$ contained in R and of the boundary of R lying above the line $y = \alpha_{i-1} - 2r - \Delta$.

The set $D^*_{\alpha i}$ consists of points s from $D_{\alpha i}$ that are a distance of b away from the horizontal boundaries of $D_{\alpha i}$ and a distance of a away from its vertical boundaries and additionally satisfy the following conditions: for any ellipse $E(X, x_j, y_j, a, b)$ from $P_{\alpha(i-1)}$, it is true that $s \notin \operatorname{int} E(X, x_j, y_j, 2a, 2b)$. Here, $P_{\alpha(i-1)}$ is the packing obtained at the preceding level. Therefore, the construction of $D^*_{\alpha i}$, $1 < i \le g$, makes use of the packing $P_{\alpha(i-1)}$ constructed for the subset $D_{\alpha(i-1)}$. Next, on $D^*_{\alpha i}$, a grid is constructed whose nodes generate a set $T_{\alpha i}$. The latter is used to construct an auxiliary problem with allowance for the level weights. Solving this problem yields a packing $P_{\alpha i}$ for the subset $D_{\alpha i}$.

For chosen grid steps, suppose that, for R, a set R^* is found, a grid is constructed on R^*, and its nodes are used to construct a set T. Assume that, on each level of the set $D_{\alpha i}$, identical grid steps are chosen and the corresponding sets $T_{\alpha i}$, $1 \le i \le g$ are constructed stage by stage. Let n be the number of elements in T

and n_i be the number of elements in $T_{\alpha i}$, $1 \le i \le g$. It is important that each n_i ($1 \le i \le g$) be less than n and, additionally, the sum of all n_i be also less than n. Thus, the dimensions of the auxiliary problems for $D_{\alpha i}$ are lower than the dimension of problem (4), (5) for the entire set R.

For vertical ellipses, the procedure is similar to that described above, while, for ellipses with differently oriented major axes, the procedure has to take into account the nonintersection conditions for horizontal and vertical ellipses and ones with mutually orthogonal major axes.

If the number n of variables (dimension) of problem (4), (5); (6), (7); or (12), (13) does not exceed n^*, then the chosen problem is regarded as solvable in an acceptable time. If $n > n^*$, then we use the following algorithm.

Algorithm

1. On the domain R, construct g ($g \ge 2$) subsets $D_{\alpha i}$, $1 \le i \le g$, such that the auxiliary problems constructed for them have an acceptable dimension.

2. Sequentially solve the auxiliary problems for each subset, starting from $D_{\alpha 1}$, to obtain packings in these subsets. The union of the resulting packings for the subsets $D_{\alpha i}$, $1 \le i \le g$, is regarded as the solution of the packing problem for R. Complete the procedure.

Obviously, the packings for individual subsets $D_{\alpha i}$, $1 \le i \le g$, are not independent. The solution for each subset, starting from the second, depends on the solutions obtained for the preceding subsets. However, even if an optimal packing is obtained for each subset, it does not mean that the resulting packing is optimal. Nevertheless, the numerical computations performed have shown that the algorithm produces acceptable results in an acceptable time. In other words, the algorithm can be viewed as effective.

The parameters $\alpha_1, \alpha_2, \ldots, \alpha_{g-1}$ of the algorithm, which determine the subsets $D_{\alpha i}$, $1 \le i \le g$, are chosen at the initial stage so that the auxiliary problems are solvable in an acceptable time. For packing horizontal ellipses, the values of $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$ can be optimized as follows. Given α_1 and a set $D_{\alpha 1}$, problem (4), (5) is solved without introducing level weights. As a result, we obtain a packing of, say, q_1 ellipses in $D_{\alpha 1}$, $D_{\alpha 1} \subseteq R$. Gradually reducing α_1, we find its smallest value (with the prescribed accuracy) at which the number of packed ellipses remains equal to q_1. Before selecting the next value α_2, for $D_{\alpha 1}$ (with chosen α_1), we need to solve an auxiliary problem with level weights, so that the ellipses are placed as low as possible, while their number remains equal to q_1. Next, α_2 is refined, etc.

The level weights for $D_{\alpha j}$, $1 \le j \le g$, are chosen according to the formula $c_{ji} = 1 + \tau(\alpha_j - y_i)$, $1 \le i \le l_j$, where c_{ji} is the weight of the ith level in $D_{\alpha j}$, τ is a tuning parameter, α_j is a parameter determining the subset $D_{\alpha j}$, y_j is the y coordinate of the ith level in $D_{\alpha j}$, l_j is the number of levels in $D_{\alpha j}$ (the levels are indexed from bottom to top), and $1 \le j \le g$. The value of τ is chosen from the interval [0.1, 0.35].

For vertical ellipses and ellipses with mixed orientations, the values of $\alpha_1, \alpha_2, \ldots, \alpha_{g-1}$ are optimized in a similar fashion to the case of horizontal ellipses.

9. NUMERICAL RESULTS

Ellipses were packed in a square Q with a side length of 3 and in a rectangle R with a width of 3 and height of 6 units. Thus, R was formed of two squares Q one placed on top of the other. The side length of the square was set to three, because, in some of the cases considered in this paper, the radius r of circles packed in Q was such that, when divided by 3, it gave the best known radius of circles packed into a unit square. Clearly, setting a radius of 0.5 for the circles to be packed in Q is more natural than setting a radius of 0.1666 (with 6 repeating) for packing in a unit square. Equal ellipses with various parameters were packed in Q and R.

To optimize packing, we developed a special software code using the library CPLEX 11.2, which makes it possible to solve problems of the indicated types. The code optimizes packings of (1) only horizontal ellipses, (2) only vertical ellipses, or (3) orthogonally oriented ellipses. In all three cases, we found the maximum possible number of ellipses with given parameters that can be packed in a given figure and determined the positions of the ellipses' centers and orientations (horizontal or vertical). Since the maximum possible number of all ellipses was found, it was possible in the third case that all ellipses were either horizontal or vertical or there were ellipses of both types.

The computations were performed on an Intel Core 2 Duo CPU T7300 2.00 GHz computer with 2 GB RAM run under Windows XP.

Table 2. Packings of ellipses in the square Q and the rectangle R

Ellipse parameters a/b	Radius of circle r	Shape to be packed in	Circle packing		Ellipse packing		
			Number of circles/density n/p	Best available packings of circles of radius $r/3$ in unit square n^*/p^*	Number of horizontal ellipses/density n_g/p_g	Number of vertical ellipses/density n_v/p_v	Number of ellipses with differently oriented axes/density
0.68892/0.45928	0.5625	Q	6/0.663	6–7/0.664–0.669	6/0.663		6/0.663
		R	13/0.718		12/0.663	12/0.663	$13 = 3g + 10v/0.718$ or $13 = 4g + 9v/0.718$
0.61237/0.40825	0.5	Q	9/0.785	9/0.785	7/0.611		$8 = 2g + 6v/0.698$
		R	18/0.785		15/0.654	16/0.698	$16 = 0g + 16v/0.698$
0.45928/0.30619	0.375	Q	16/0.785	16/0.785	14/0.657		$14 = 14g + 0v/0.687$
		R	32/0.785		28/0.687	30/0.736	$30 = 0g + 30v/0.736$
0.38273/0.25515	0.3125	Q	22/0.750	24–25/0.775–0.785	21/0.716		$21 = 21g + 0v/0.736$
		R	45/0.767		45/0.767	45/0.767	$45 = 24g + 21v/0.767$
0.33681/0.22454	0.275	Q	28/0.739	30/0.792	27/0.713		$28 = 11g + 17v/0.739$
		R	61/0.805		54/0.713	56/0.739	$56 = 0g + 56v/0.739$
0.30619/0.20412	0.25	Q	36/0.785	36/0.785	31/0.676		$33 = 4g + 29v/0.719$ or $33 = 16g + 17v/0.753$
		R	74/0.807		69/0.75	65/0.71	$69 = 69g + 0v/0.753$

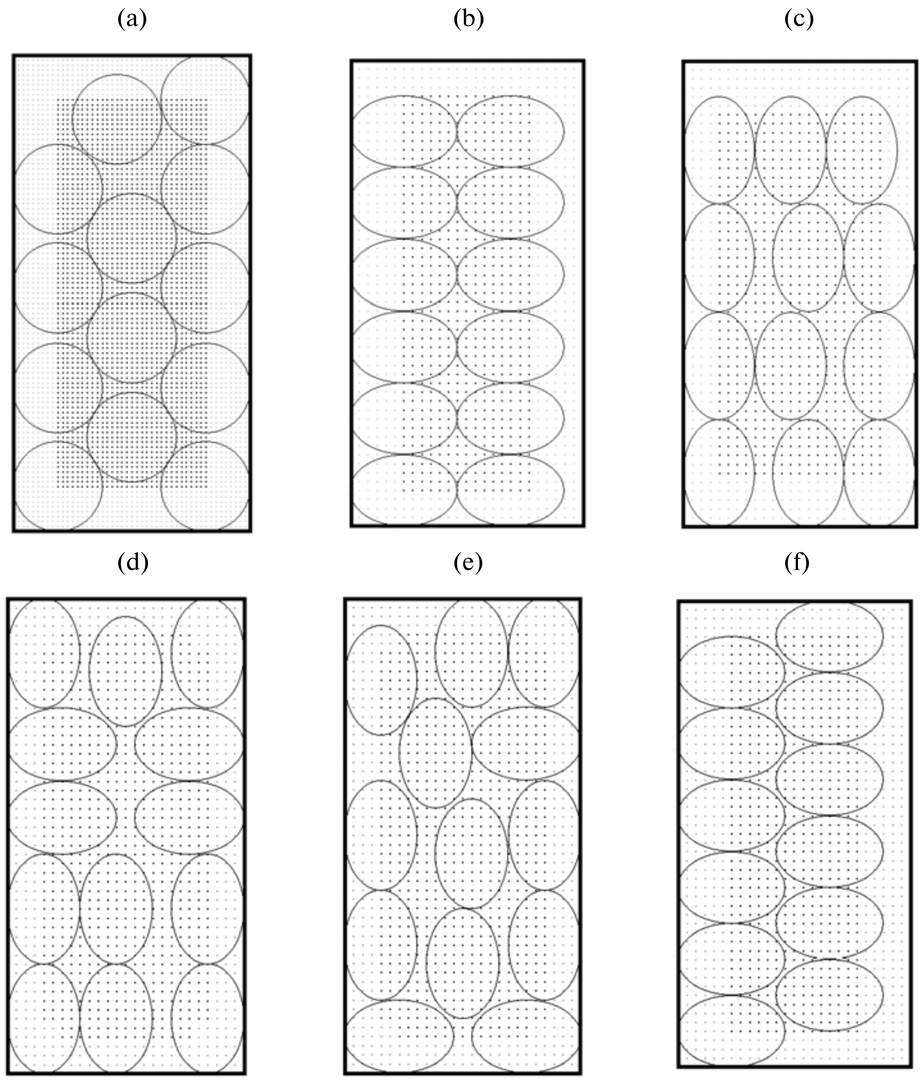

Fig. 3. Packings in R of (a) 13 circles, (b) 12 horizontal ellipses pressed down, (c) 12 vertical ellipses, (d) 4 horizontal and 9 vertical ellipses, (e) 3 horizontal and 10 vertical ellipses, and (f) 12 horizontal ellipses pressed up or down.

The resulting packings of ellipses were evaluated as follows. For a given r, we packed the maximum possible number of equal circles of radius r in Q. The resulting packing was compared with the best available packing of circles of radius $r/3$ in a unit square. The best available packing densities for circles in a unit square were denoted by p^*, while the known largest numbers of packed circles were denoted by n^*. For a given ellipse eccentricity, the ellipse parameters a and b were calculated so that the area of the ellipse coincided with the area of a circle of radius r. Next, the packing densities for ellipses and circles (with identical areas) were compared.

Table 2 presents the numerical results obtained for Q and R with various ellipse parameter at the fixed eccentricity $e = 0.74536$. Here, the slash symbol (/) is used to separate the results. The following characteristics are given In Table 2:

- The first column contains the semimajor and semiminor axes of the ellipses.
- The second column gives the radii of circles having the same area as that of an ellipse with parameters indicated in the given line.
- The third column specifies the figures into which ellipses and circles were packed; the results for Q and R are given in the corresponding lines.
- The fourth column contains the resulting numbers of packed circles of radius r and the corresponding packing densities (n/p).

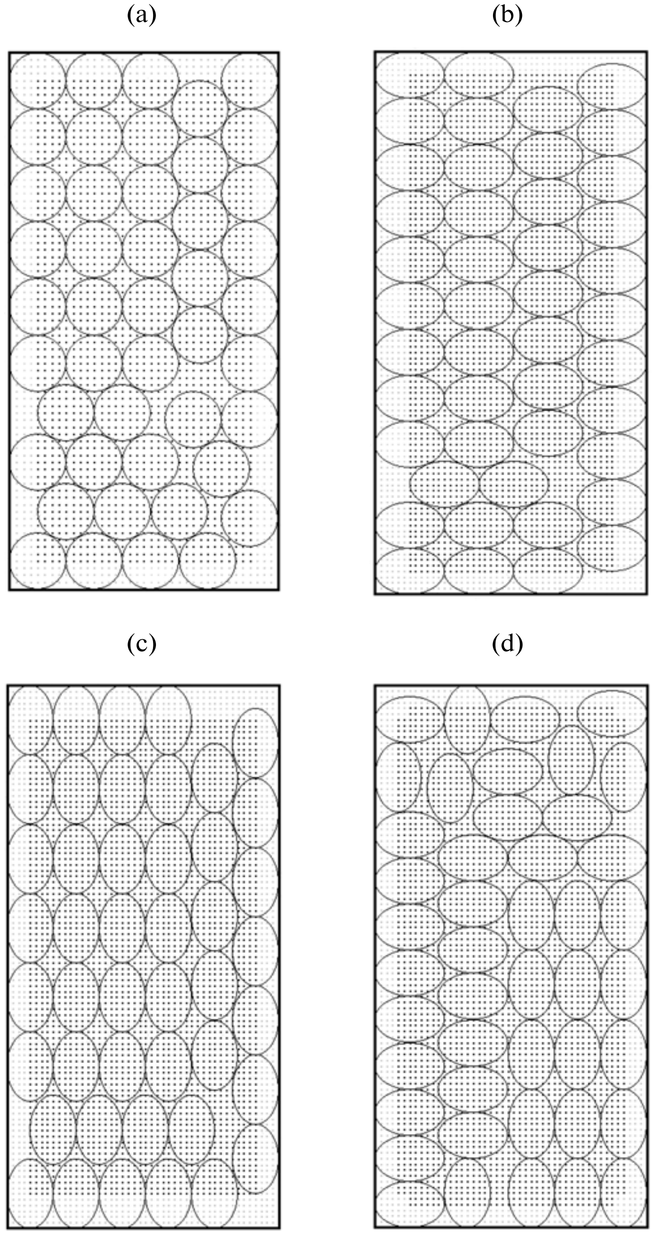

Fig. 4. Packings in R of (a) 45 circles, (b) 45 horizontal ellipses, (c) 45 vertical ellipses, and (d) 24 horizontal and 21 vertical ellipses.

• The fifth column gives the known maximum possible numbers of circles of radius $r/3$ that can be packed in a unit square and the corresponding packing densities (n^*/p^*).

• The sixth column presents the resulting numbers (n_g) of packed horizontal ellipses and the corresponding packing density (p_g).

• The seventh column contains the resulting numbers (n_v) of packed vertical ellipses and the corresponding packing densities (p_v); for packings in Q, the results of this column coincide with those of the preceding column and, for this reason, are omitted.

• The eighth column gives the resulting numbers of packed orthogonally oriented ellipses and the corresponding packing densities.

Note that the numbers of packed ellipses obtained with and without the heuristic (in the latter case, if that was possible on our computer) turned out to be identical, but the CPU times were usually widely different.

Inspection of Table 2 shows that the resulting numbers of circles packed in Q (and, hence, the packing density) coincide with the best known results (see Table 13.1 in [5]), except for $r = 0.275$ and 0.3125. Moreover, the best known results for $r = 0.5$, 0.375 and 0.25 are optimal (unimprovable) (see, e.g., [5]).

The number of circles packed in R is everywhere two and more times larger than the number of circles packed in Q. Based on this result, we conclude that the resulting packings in R are acceptable. The number of ellipses packed in Q is, as a rule, less than the number of circles (of equal area). For ellipses packed in R, the numbers of packed ellipses and circles coincide in half of the cases.

The packings of circles in Q were obtained (on the above-mentioned computer) by solving system (4), (5) without applying the heuristic at radii $r \geq 0.275$. Starting at $r = 0.25$, Q was divided into two parts. In the case of ellipse packing in the square, the heuristic was not used for ellipse parameters corresponding to $r \geq 0.375$ and was used for $r = 0.275$ and 0.25.

For circles packed in R, system (4), (5) was solved without applying the heuristic for $r \geq 0.375$, while, for $r = 0.275$ and 0.25, the heuristic was used. For ellipses packed in R, the heuristic was not used for $r =$ 0.5625 and 0.5, while being used for the other ellipse parameters.

In the case where the radius of circles is $r = 0.5625$, it was found that 13 circles can be packed in R (see Fig. 3a). Packings of 12 ellipses were obtained for vertical and horizontal ellipses (see Figs. 3b and 3c, respectively). In the case of both horizontal and vertical ellipses, 13 ellipses can be packed in R with the packing density equal to that for 13 circles. Figures 3d and 3e show packings of 13 ellipses with various ratios of the numbers of horizontal and vertical ellipses.

The introduction of level weights and the pressing of ellipses were found to be useful. More specifically, Fig. 3b shows the maximum possible number of horizontal ellipses packed in R as obtained with level weights and the pressing down procedure, while Fig. 3e displays similar results obtained without level weights. The number of ellipses is identical in both cases. From a practical point of view, however, the packing in Fig. 3b can be more useful, since it leaves a single large piece in the upper part of R intact, in contrast to the case depicted in Fig. 3e. This large piece of the rectangle (material) can possibly be used for something else. When a given number of ellipses are cut off from a strip, the level weights make it possible to minimize the length of the strip used. The result of packing can depend on whether the ellipses are pressed up or down, right or left. The packings in R were obtained by pressing the ellipses down via choosing suitable level weights.

Figure 4 presents the resulting packings of (a) 45 circles, (b) 45 horizontal ellipses, (c) 45 vertical ellipses, and (d) 24 horizontal ellipses and 21 vertical ellipses. The packing densities p are identical in all these cases. Specifically, $p = 0.767$, which differs from the best known packing density for circles packed in the square by at most 0.02.

In the case of packing both horizontal and vertical ellipses, in each considered case (presented in Table 2), the number of ellipses was no less (and sometimes more) than that for entirely horizontal or vertical ellipses.

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